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Color degree and color neighborhood union conditions for long heterochromatic paths in edge-colored graphs *

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Abstract

Let G be an edge-colored graph. A heterochromatic (rainbow, or multi-colored) path of G is such a path in which no two edges have the same color. Let $d^c(v)$ denote the color degree and $CN(v)$ denote the color neighborhood of a vertex v of G. In a previous paper, we showed that if $d^c(v) \geq k$ (color degree condition) for every vertex v of G, then G has a heterochromatic path of length at least $\lceil \frac{k+1}{2} \rceil$, and if $|CN(u) \cup CN(v)| \geq s$ (color neighborhood union condition) for every pair of vertices u and v of G, then G has a heterochromatic path of length at least $\lceil \frac{s}{3} \rceil + 1$. Later, in another paper we first showed that if $k \leq 7$, G has a heterochromatic path of length at least $k-1$, and then, based on this we use induction on k and showed that if $k \geq 8$, then G has a heterochromatic path of length at least $\lceil \frac{3k}{5} \rceil + 1$. In the present paper, by using a simpler approach we further improve the result by showing that if $k \geq 8$, G has a heterochromatic path of length at least $\lceil \frac{2k}{3} \rceil + 1$, which confirms a conjecture by Saito. We also improve a previous result by showing that under the color neighborhood union condition, G has a heterochromatic path of length at least $\lfloor \frac{2s+4}{5} \rfloor$.

Keywords: edge-colored graph, color degree, color neighborhood, heterochromatic (rainbow, or multicolored) path.

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1. Introduction

We use Bondy and Murty [3] for terminology and notations not defined here and consider simple graphs only.

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Let $G = (V, E)$ be a graph. By an *edge-coloring* of G we will mean a function $C : E \to \mathbb{N}$, the set of natural numbers. If G is assigned such a coloring, then we say that G is an *edge-colored graph.* Denote the colored graph by (G, C), and call $C(e)$ the *color* of the edge $e \in E$. We say that $C(uv) = \emptyset$ if $uv \notin E(G)$ for $u, v \in V(G)$. For a subgraph H of G, we denote $C(H) = \{C(e) \mid e \in E(H)\}$ and $c(H) = |C(H)|$. For a vertex v of G, the *color neighborhood* $CN(v)$ of v is defined as the set $\{C(e) \mid e \text{ is incident with } v\}$ and the *color degree* is $d^c(v) = |CN(v)|$. A path is called *heterochromatic (rainbow, or multicolored)* if any two edges of it have different colors. If u and v are two vertices on a path P, uPv denotes the segment of P from u to v, whereas $vP^{-1}u$ denotes the same segment but from v to u.

There are many existing literature dealing with the existence of paths and cycles with special properties in edge-colored graphs. In [6], the authors showed that for a 2-edge-colored graph G and three specified vertices x, y and z, to decide whether there exists a color-alternating path from x to y passing through z is NP-complete. The heterochromatic Hamiltonian cycle or path problem was studied by Hahn and Thomassen [10], Rödl and Winkler (see [9]), Frieze and Reed [9], and Albert, Frieze and Reed [1]. For more references, see [2, 7, 8, 11, 12]. Many results in these papers were proved by using probabilistic methods.

In [4], the authors showed that if G is an edge-colored graph with $d^c(v) \geq k$ (color degree condition) for every vertex v of G, then G has a heterochromatic path of length at least $\lceil \frac{k+1}{2} \rceil$, and if $|CN(u) \cup CN(v)| \geq s$ (color neighborhood union condition) for every pair of vertices u and v of G, then G has a heterochromatic path of length at least $\lceil \frac{s}{3} \rceil + 1$. In [5], we first showed that if $3 \leq k \leq 7$, G has a heterochromatic path of length at least $k - 1$, and then, based on this we use induction on k and showed that if $k \geq 8$, then G has a heterochromatic path of length at least $\lceil \frac{3k}{5} \rceil + 1$. In the present paper, by using a simpler approach we further improve the result by showing that if $k \geq 8$, G has a heterochromatic path of length at least $\lceil \frac{2k}{3} \rceil + 1$, which confirms a conjecture by Saito. We also show that under the color neighborhood union condition, G has a heterochromatic path of length at least $\lfloor \frac{2s+4}{5} \rfloor$.

2. Long heterochromatic paths under the color degree condition

In this section we will give a better lower bound for the length of the longest heterochromatic path in G when $k \geq 8$. As an induction initial, we need the following result as a lemma.

Lemma 2.1 ([5]) *Let G be an edge-colored graph and $3 \leq k \leq 7$ an integer. Suppose that $d^c(v) \geq k$ for every vertex v of G. Then G has a heterochromatic path of length at least $k - 1$.*

Then, we need to do the following preparations.

Lemma 2.2 *Suppose $P = u_1u_2\dots u_lu_{l+1}$ is a longest heterochromatic path. If there exists an x such that $3 \le x \le l$ and $C(u_1u_x) \notin C(P)$, then $C(u_{x-1}u_x) \notin (CN(u_{l+1}) - C(u_1u_{l+1}, u_2u_{l+1}, \dots, u_{l-1}u_{l+1}))$.*

Proof. By contradiction. If there exists a $v \in V(G) - V(P)$ such that $C(u_{l+1}v) = C(u_{x-1}u_x)$, then $u_{x-1}P^{-1}u_1u_xPu_{l+1}v$ is a heterochromatic path of length $l+1$, a contradiction. So $C(u_{x-1}u_x) \notin (CN(u_{l+1}) - C(u_1u_{l+1}, u_2u_{l+1}, \dots, u_{l-1}u_{l+1}))$. ∎

Lemma 2.3 *Suppose $P = u_1u_2\dots u_lu_{l+1}$ is a longest heterochromatic path, $v \in V(G) - V(P)$ and $C(u_{l+1}v) = C(u_1u_2)$. If there exists an x such that $2 \le x \le l-2$ and $|C(u_xv, u_{x+2}v) - C(P)| = 2$, then $C(u_xu_{x+1}, u_{x+1}u_{x+2}) \cap (CN(u_{l+1}) - C(u_1u_{l+1}, u_2u_{l+1}, \dots, u_{l-1}u_{l+1})) = \emptyset$.*

Proof. By contradiction. If there exists a $v' \in V(G) - V(P)$ such that $u_{l+1}v' \in E(G)$ and $C(u_{l+1}v') \in C(u_xu_{x+1}, u_{x+1}u_{x+2})$, then $u_1Pu_xvu_{x+2}Pu_{l+1}v'$ is a heterochromatic path of length $l+1$, a contradiction. So $C(u_xu_{x+1}, u_{x+1}u_{x+2}) \cap (CN(u_{l+1}) - C(u_1u_{l+1}, u_2u_{l+1}, \dots, u_{l-1}u_{l+1})) = \emptyset$. ∎

Lemma 2.4 *Let $P = u_1u_2\dots u_lu_{l+1}v_1$ be a path in G such that*

(a) u_1Pu_{l+1} is a longest heterochromatic path in G;

(b) $C(u_{l+1}v_1) = C(u_{j_0}u_{j_0+1})$ and $1 \le j_0 \le l$ is as small as possible, subject to (a).

Then $C(u_1u_{j_0+1}, u_1u_{j_0+2}, \dots, u_1u_{2j_0}) \subseteq C(P)$.

Proof. By contradiction. If there exists an x such that $j_0 + 1 \le x \le 2j_0$ and $C(u_1u_x) \notin C(P)$, then $u_{x-1}P^{-1}u_1u_xPu_{l+1}$ is a heterochromatic path of length l and $u_{j_0+1}u_{j_0}$ is the $x - j_0 - 1 \le 2j_0 - j_0 - 1 = j_0 - 1 < j_0$-th edge in this heterochromatic path, contradicting the choice of P. Therefore $C(u_1u_{j_0+1}, u_1u_{j_0+2}, \dots, u_1u_{2j_0}) \subseteq C(P)$. ∎

Lemma 2.5 *Let $P = u_1u_2\dots u_lu_{l+1}v_1$ be a path in G such that*

(a) u_1Pu_{l+1} is a longest heterochromatic path in G;

(b) $C(u_{l+1}v_1) = C(u_{j_0}u_{j_0+1})$ and $1 \le j_0 \le l$ is as small as possible, subject to (a).

Then for any $2j_0 + 1 \le x \le l$, $|C(u_1u_x, u_1u_{x+1}) - C(P)| \le 1$.

Proof. By induction. If there exists an x such that $2j_0 + 1 \le x \le l$ and $|C(u_1u_x, u_1u_{x+1}) - C(P)| = 2$, then $u_2Pu_xu_1u_{x+1}Pu_{l+1}$ is a heterochromatic path of length l and $u_{j_0}u_{j_0+1}$ is the $(j_0 - 1)$-th edge in this heterochromatic path, contradicting the choice of P. Therefore $|C(u_1u_x, u_1u_{x+1}) - C(P)| \le 1$ for any $2j_0 + 1 \le x \le l$. ∎

Lemma 2.6 *Suppose $d^c(v) \geq k$ for every vertex $v \in V(G)$ and the length of a longest heterochromatic path in G is $l = \lceil \frac{2k}{3} \rceil$. Then there is a heterochromatic path $P = u_1u_2 \ldots u_l u_{l+1}$ in G and a $v \in V(G) - V(P)$ such that $C(u_{l+1}v) = C(u_1u_2)$.*

Proof. Let $P = u_1u_2 \ldots u_l u_{l+1} v_1$ be a path in G such that

(a) $u_1 P u_{l+1}$ is a longest heterochromatic path in G;

(b) $C(u_{l+1}v_1) = C(u_{j_0}u_{j_0+1})$ and $1 \leq j_0 \leq l$ is as small as possible, subject to (a).

Then we claim that $j_0 = 1$. We will show this by contradiction. Suppose $j_0 > 1$. Denote $i_j = C(u_j u_{j+1})$ for $1 \leq j \leq l$.

Since the longest heterochromatic path in G is of length l, for any $v \in V(G) - V(u_1 P u_{l+1})$ we have $C(u_1 v) \in C(P)$. On the other hand, $d^c(u_1) \geq k$. So there are at least $k - l = \lfloor \frac{k}{3} \rfloor$ different colors not in $C(P)$ appearing in $\{u_1u_3, u_1u_4, \ldots, u_1u_l, u_1u_{l+1}\}$. Then there are x_i's such that $3 \leq x_1 < x_2 < \ldots < x_{k-l} \leq l+1$ and $|C(\{u_1u_{x_1}, u_1u_{x_2}, \ldots, u_1u_{x_{k-l}}\}) - C(P)| = k - l$. Therefore, by Lemma 2.2 and the assumption that $j_0 > 1$ we have $(CN(u_{l+1}) - C(u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}, u_lu_{l+1})) \subseteq C(P) - \{i_1, i_{x_1-1}, \ldots, i_{x_{k-l}-1}\}$. Since $d^c(u_{l+1}) \geq k$, we have $\lceil \frac{2k}{3} \rceil = l \geq |C(u_1u_{l+1}, u_2u_{l+1}, \ldots, u_lu_{l+1})| \geq k - |C(P) - \{i_1, i_{x_1-1}, \ldots, i_{x_{k-l}-1}\}| = k - (l - k + l - 1) = 2k - 2l + 1 = 2\lfloor \frac{k}{3} \rfloor + 1$. Since if $k \equiv 0 \ (mod\ 3)$ then $2\lfloor \frac{k}{3} \rfloor + 1 > \lceil \frac{2k}{3} \rceil$, we need only to consider the cases when $k \equiv 1 \ (mod\ 3)$ or $k \equiv 2 \ (mod\ 3)$.

Case 1. $k \equiv 1 \ (mod\ 3)$.

In this case, we have $\lceil \frac{2k}{3} \rceil = 2\lfloor \frac{k}{3} \rfloor + 1$. Then $CN(u_{l+1}) - C(u_1u_{l+1}, u_2u_{l+1}, \ldots, u_lu_{l+1}) = C(P) - \{i_1, i_{x_1-1}, \ldots, i_{x_{k-l}-1}\}$ and $C(u_lu_{l+1}) \notin C(P) - \{i_1, i_{x_1-1}, \ldots, i_{x_{k-l}-1}\}$. Then, we can get $C(u_lu_{l+1}) = i_{x_{k-l}-1}$, i.e., $x_{k-l} = l+1$.

Noticing that $C(u_{l+1}v_1) = i_{j_0}$ and j_0 is as small as possible, we have $\{i_2, \ldots, i_{j_0-1}\} \cap (C(P) - \{i_1, i_{x_1-1}, \ldots, i_{x_{k-l}-1}\}) = \emptyset$, and so $\{3, 4, \ldots, j_0\} \subseteq \{x_1, x_2, \ldots, x_{k-l}\}$. Hence, by Lemmas 3.3 and 3.4 we have $\{x_1, x_2, \ldots, x_{k-l}\} \subseteq \{3, 4, \ldots, j_0\} \cup \{2j_0 + 1, \ldots, l+1\}$, and $|\{x_1, x_2, \ldots, x_{k-l}\} \cap \{2j_0 + 1, 2j_0 + 2, \ldots, l+1\}| \leq \lfloor \frac{(l+1)-(2j_0+1)}{2} \rfloor + 1 = \lfloor \frac{l}{2} \rfloor - j_0 + 1$. Consequently, $|\{x_1, x_2, \ldots, x_{k-l}\}| \leq (j_0 - 2) + \lfloor \frac{l}{2} \rfloor - j_0 + 1 = \lfloor \frac{l}{2} \rfloor - 1 < \lfloor \frac{k}{3} \rfloor = k - l$, a contradiction.

Case 2. $k \equiv 2 \ (mod\ 3)$.

In this case, we have $\lceil \frac{2k}{3} \rceil = (2\lfloor \frac{k}{3} \rfloor + 1) + 1$. We distinguish the following two cases:

Case 2.1. $x_{k-l} = l + 1$.

Since $\{x_1, x_2, \ldots, x_{k-l}\} \subseteq \{3, 4, \ldots, j_0\} \cup \{2j_0+1, \ldots, l+1\}$ by Lemma 2.4, and $|\{x_1, x_2, \ldots, x_{k-l}\} \cap \{3, 4, \ldots, j_0\}| \leq |\{3, 4, \ldots, j_0\}| = j_0 - 2$, $|\{x_1, x_2, \ldots, x_{k-l}\} \cap \{2j_0 + 1, \ldots, l+1\}| \leq \lfloor \frac{(l+1)-(2j_0+1)}{2} \rfloor + 1 = \lfloor \frac{l}{2} \rfloor - j_0 + 1$ by Lemma 2.5, we have $|\{x_1, x_2, \ldots, x_{k-l}\}| \leq (j_0 - 2) + (\lfloor \frac{l}{2} \rfloor - j_0 + 1) = \lceil \frac{l}{2} \rceil - 1 = \lfloor \frac{k}{3} \rfloor = k - l$. Then $\{x_1, x_2, \ldots, x_{k-l}\} = \{3, 4, \ldots, j_0, 2j_0 + 1, 2j_0 + 3, \ldots, l-1, l+1\}$.

Since $\lceil \frac{2k}{3} \rceil = (2\lfloor \frac{k}{3} \rfloor + 1) + 1$, there is at most one color in $C(P) - \{i_1, i_{x_1-1}, \ldots, i_{x_{k-l}-1}\}$ contained in $C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_lu_{l+1}\})$, i.e., $|(C(P) - \{i_1, i_{x_1-1}, \ldots, i_{x_{k-l}-1}\}) - (CN(u_{l+1}) - C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_lu_{l+1}\}))| \leq 1$.

If $j_0 \geq 3$, then $|\{j_0+1, \ldots, 2j_0-1\}| = j_0 - 1 \geq 2$, and so there exists a $v \in V(G) - V(P)$ such that $C(u_{l+1}v) = i_{j_0+s}$ for some $1 \leq s \leq j_0 - 1$. Then $u_{2j_0}P^{-1}u_1u_{2j_0+1}Pu_{l+1}$ is a heterochromatic path of length l and $u_{j_0+s+1}u_{j_0+s}$ is the $(j_0 - s)$-th edge in this heterochromatic path, contradicting the choice of P.

Therefore we need only to consider the case when $j_0 = 2$, then $x_1 = 2j_0+1 = 5$. In this case, there exists a $v \in V(G) - V(P)$ such that $C(u_{l+1}v) \in \{i_{j_0+1}, i_{x_1}\}$. If $C(u_{l+1}v) = i_{j_0+1} = i_{2j_0-1}$, then $u_{2j_0}u_{2j_0-1}$ is the first edge in the heterochromatic path $u_{2j_0}P^{-1}u_1u_{2j_0+1}Pu_{l+1}$ of length l; if $C(u_{l+1}v) = i_{x_1}$, then $u_{x_1+1}u_{x_1}$ is the first edge in the heterochromatic path $u_{x_1+1}P^{-1}u_1u_{x_2}Pu_{l+1}$ of length l, contradicting the choice of P.

Case 2.2. $x_{k-l} < l + 1$.

In this case, we can get $\{x_1, x_2, \ldots, x_{k-l}\} \subseteq \{3, 4, \ldots, j_0\} \cup \{2j_0+1, 2j_0+2, \ldots, l\}$ by Lemma 3.3, $|\{x_1, x_2, \ldots, x_{k-l}\} \cap \{3, 4, \ldots, j_0\}| \leq j_0 - 2$ and $|\{x_1, x_2, \ldots, x_{k-l}\} \cap \{2j_0+1, \ldots, l\}| \leq \lfloor \frac{l-(2j_0+1)}{2} \rfloor + 1 = \lfloor \frac{l-1}{2} \rfloor - j_0 + 1 = \frac{l}{2} - j_0$ by Lemma 3.4. Consequently, $|\{x_1, x_2, \ldots, x_{k-l}\}| \leq (j_0 - 2) + (\frac{l}{2} - j_0) = \frac{l}{2} - 2 = \lfloor \frac{k}{3} \rfloor - 1 < k - l$, a contradiction.

From the arguments of all the above cases, we get that j_0 cannot be larger than 1, and so $j_0 = 1$. ∎

Now we are ready to give our main result.

Theorem 2.7 *If $d^c(v) \geq k \geq 7$ for any $v \in V(G)$, then G has a heterochromatic path of length at least $\lceil \frac{2k}{3} \rceil + 1$.*

Proof. We will prove the theorem by induction.

If $k = 7$, our Lemma 2.1 guarantees that G has a heterochromatic path of length at least $6 = \lceil \frac{2\times 7}{3} \rceil + 1$.

Assume that if $d^c(v) \geq k-1$ for any $v \in V(G)$, G has a heterochromatic path of length at least $\lceil \frac{2(k-1)}{3} \rceil + 1$. Then we need only to show that if $d^c(v) \geq k$ for any $v \in V(G)$, G has a heterochromatic path of length $\lceil \frac{2k}{3} \rceil + 1$. Since if $k \equiv 0$ $(mod\ 3)$ then $\lceil \frac{2(k-1)}{3} \rceil + 1 = \lceil \frac{2k}{3} \rceil + 1$, we need only to show that if $k \equiv 1, 2$ $(mod\ 3)$, G has a heterochromatic path of length at least $\lceil \frac{2k}{3} \rceil + 1$.

By the assumption we know that G has a heterochromatic path of length at least $\lceil \frac{2(k-1)}{3} \rceil + 1 = \lceil \frac{2k}{3} \rceil$. Assume that the longest heterochromatic path in G is of length $\lceil \frac{2k}{3} \rceil$. Then, by Lemma 2.6 G has a heterochromatic path $P = u_1u_2 \ldots u_lu_{l+1}$ of length $l = \lceil \frac{2k}{3} \rceil$ and there exists a $v_1 \in V(G) - V(P)$ such that $C(u_{l+1}v_1) = C(u_1u_2)$. Denote $i_j = C(u_ju_{j+1})$ for $1 \leq j \leq l$.

Since $d^c(v_1) \geq k$, we have that $d^c(u_1) \geq k$ and the longest heterochromatic path in G is of length l, and so there exist y_i's and x_j's such that $2 \leq y_1 < y_2 < y_3 < \ldots < y_{k-l} \leq l$ and $3 \leq x_1 < x_2 < \ldots < x_{k-l} \leq l+1$, and $|C(\{u_{y_1}v_1, u_{y_2}v_1, \ldots, u_{y_{k-l}}v_1\}) - C(P)| = k - l$ and $|C(\{u_1u_{x_1}, u_1u_{x_2}v_1, \ldots, u_1u_{x_{k-l}}\}) - C(P)| = k - l$.

If there exists a j_0 such that $1 \le j_0 \le k-l-1$ and $y_{j_0+1} = y_{j_0}+1$, then $u_1Pu_{y_{j_0}}v_1u_{y_{j_0}+1}Pu_{l+1}$ is a heterochromatic path of length $l+1$, a contradiction.

If $y_1 = 2$, since $k-l = \lfloor \frac{k}{3} \rfloor \ge 2$, then there exists a j such that $1 \le j \le k-l$ and $C(u_1u_{x_j}) \notin C(P) \cup C(u_2v_1)$. Then $u_{x_j-1}P^{-1}u_2v_1u_{l+1}P^{-1}u_{x_j}u_1$ is a heterochromatic path of length $l+1$, a contradiction.

If $x_{k-l} = l+1$, since $k-l = \lfloor \frac{k}{3} \rfloor \ge 2$, then there exists a j such that $1 \le j \le k-l$ and $C(u_{y_j}v_1) \notin (C(P) \cup C(u_1u_{l+1}))$. Then $v_1u_{y_j}P^{-1}u_1u_{l+1}P^{-1}u_{y_j+1}$ is a heterochromatic path of length $l+1$, a contradiction.

If there exists a j_0 such that $1 \le j_0 \le k-l-1$ and $x_{j_0+1} = x_{j_0}+1$, then $u_2Pu_{x_{j_0}}u_1u_{x_{j_0}+1}Pu_{l+1}v_1$ is a heterochromatic path of length $l+1$, a contradiction.

Consequently, we have $3 \le x_1 < x_1+1 < x_2 < x_2+1 < \ldots < x_{k-l} \le l$ and $3 \le y_1 < y_1+1 < y_2 < y_2+1 < \ldots < y_{k-l} \le l$. Then $CN(u_{l+1}) - C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}, u_lu_{l+1}\}) \subseteq C(P) - \{i_{x_1-1}, i_{x_2-1}, \ldots, i_{x_{k-l}-1}\}$ by Lemma 2.2 and the fact that P is the longest heterochromatic path in G. On the other hand, $x_{k-l} \le l$, and so $i_l \in C(P) - \{i_{x_1-1}, i_{x_2-1}, \ldots, i_{x_{k-l}-1}\}$. Then $CN(u_{l+1}) - C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}\}) \subseteq C(P) - \{i_{x_1-1}, i_{x_2-1}, \ldots, i_{x_{k-l}-1}\}$.

We distinguish the following two cases:

Case 1. $k \equiv 1$ $(mod\ 3)$.

Since $d^c(u_{l+1}) \ge k$, we have $l-1 \ge |C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}\})| \ge k - |C(P) - \{i_{x_1-1}, i_{x_2-1}, \ldots, i_{x_{k-l}-1}\}| = k-l+(k-l) = l-1$. Therefore $u_1u_{l+1} \in E(G)$ and $C(u_1u_{l+1}) \in \{i_{x_1-1}, i_{x_2-1}, \ldots, i_{x_{k-l}-1}\}$. Suppose $C(u_1u_{l+1}) = C(u_{x_j-1}u_{x_j})$ for some $1 \le j \le k-l$.

On the other hand, since $3 \le y_1 < y_1+1 < y_2 < y_2+1 < \ldots < y_{k-l} \le l$ and $3 \le x_1 < x_1+1 < x_2 < x_2+1 < \ldots < x_{k-l} \le l$, we get that $2(k-l)-2 \le y_{k-l} - y_1 \le l-3 = 2(k-l)-2$ and $2(k-l)-2 \le x_{k-l} - x_1 \le l-3 = 2(k-l)-2$, and then $\{y_1, y_2, \ldots, y_{k-l}\} = \{3, 5, \ldots, l-2, l\} = \{x_1, x_2, \ldots, x_{k-l}\}$, $v_1u_{x_j}Pu_{l+1}u_1Pu_{x_j-1}$ is a heterochromatic path of length $l+1$, a contradiction.

Case 2. $k \equiv 2$ $(mod\ 3)$.

Since $3 \le y_1 < y_1+1 < y_2 < y_2+1 < \ldots < y_{k-l} \le l$, we have $2(k-l-1) \le y_{k-l} - y_1 \le l-3 = 2(k-l-1)+1$. Then we get that $y_{j+1} = y_j+2$ for $j = 1, 2, \ldots, k-l-1$ or there exists a j_0 such that $1 \le j_0 \le k-l-1$, and $y_{j+1} = y_j+2$ for any $1 \le j \le k-l-1$ and $j \ne j_0$, $y_{j_0+1} = y_{j_0}+3$.

Case 2.1 $y_{j+1} = y_j+2$ for $j = 1, 2, \ldots, k-l-1$.

In this case, we have $(CN(u_{l+1}) - C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}, u_lu_{l+1}\})) \subseteq C(P) - \{i_{y_1}, i_{y_1+1}, i_{y_2}, i_{y_2+1}, \ldots, i_{y_{k-l}-1}\}$ by Lemma 2.3 and the fact that P is the longest heterochromatic path in G. Noticing that $y_{k-l} \le l$, we have $i_l \notin \{i_{y_1}, i_{y_1+1}, i_{y_2}, i_{y_2+1}, \ldots, i_{y_{k-l}-1}\}$. Then $(CN(u_{l+1}) - C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}\})) \subseteq C(P) - \{i_{y_1}, i_{y_1+1}, i_{y_2}, i_{y_2+1}, \ldots, i_{y_{k-l}-1}\}$.

On the other hand, $CN(u_{l+1}) - C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}\}) \subseteq C(P) - \{i_{x_1-1}, i_{x_2-1}, \ldots, i_{x_{k-l}-1}\}$. Therefore $(CN(u_{l+1}) - C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}\})) \subseteq C(P) - \{i_{y_1}, i_{y_1+1}, i_{y_2}, i_{y_2+1}, \ldots, i_{y_{k-l}-1}\} \cup \{i_{x_1-1}, i_{x_2-1}, \ldots, i_{x_{k-l}-1}\}$.

Note that $3 \le x_1 < x_1+1 < x_2 < \ldots < x_{k-l} \le l$. Then $\{x_1, x_2, \ldots, x_{k-l}\} -$

$\{y_1+1, y_2, y_2+1, \ldots, y_{k-l}\} \neq \emptyset$ and $\{i_{x_1-1}, i_{x_2-1}, \ldots, i_{x_{k-l}-1}\} - \{i_{y_1}, i_{y_1+1}, i_{y_2}, \ldots, i_{y_{k-l}-1}\} \neq \emptyset$.

Consequently, $l-1 \geq |C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}\})| \geq k - |C(P) - \{i_{y_1}, i_{y_1+1}, i_{y_2}, i_{y_2+1}, \ldots, i_{y_{k-l}-1}\} \cup \{i_{x_1-1}, i_{x_2-1}, \ldots, i_{x_{k-l}-1}\}| \geq k-l+2(k-l-1)+1 = 3k-3l-1$. It is easy to check that if $k > 8$, $3k-3l-1 > l-1$, and so we need only to consider the case when $k = 8$.

If $k = 8$, $l-1 = 3k-3l-1$, and so we need only to consider the case when $|\{i_{x_1-1}, i_{x_2-1}\} - \{i_{y_1}, i_{y_1+1}\}| = 1$. Denote $i_7 = C(u_{y_1}v_1)$, $i_8 = C(u_{y_2}v_1)$. We distinguish the following two cases:

Case 2.1.1 $y_1 = 3$ and $y_2 = 5$.

In this case, we need only to consider the cases when $x_1 = 3$ and $x_2 = 5$, or $x_1 = 4$ and $x_2 = 6$. Then $C(u_1u_7) \in \{i_2, i_3, i_4\}$ or $C(u_1u_7) \in \{i_3, i_4, i_5\}$. If $C(u_1u_7) = i_3$ or i_5, then $u_4u_5v_1u_3u_2u_1u_7u_6$ is a heterochromatic path of length 7; if $C(u_1u_7) = i_2$ or i_4, then $u_4u_3v_1u_5u_6u_7u_1u_2$ is a heterochromatic path of length 7, a contradiction.

Case 2.1.2 $y_1 = 4$ and $y_2 = 6$.

In this case, we need only to consider the cases when $x_1 = 3$ and $x_2 = 5$, or $x_1 = 3$ and $x_2 = 6$, or $x_1 = 4$ $x_2 = 6$. Then $C(u_1u_7) \in \{i_2, i_4, i_5\}$ or $\{i_3, i_4, i_5\}$. If $C(u_1u_7) = i_3$ or i_5, then $u_5u_4v_1u_6u_7u_1u_2u_3$ is a heterochromatic path of length 7; if $C(u_1u_7) = i_4$, then $u_5u_6v_1u_4u_3u_2u_1u_7$ is a heterochromatic path of length 7. So, we may assume $C(u_1u_7) = i_2$. Then $C(u_2u_7, u_3u_7, u_4u_7, u_5u_7) \cap \{i_1, i_3, i_4, i_5, i_6\} \subseteq \{i_4, i_5\}$ and $|\{C(u_2u_7, u_3u_7, u_4u_7, u_5u_7)\}| = 4$. So $C(u_3u_7) = i_4$ or i_5 or some color $\notin \{i_1, i_2, \ldots, i_6\}$. Let

$$P' = \begin{cases} v_1u_4u_5u_6u_7u_3u_2u_1 & \text{if } C(u_3u_7) \notin \{i_1, i_2, \ldots, i_6, i_7\}; \\ u_2u_1u_7u_3u_4u_5u_6v_1 & \text{if } C(u_3u_7) = i_7\ ; \\ u_2u_1u_7u_3u_4v_1u_6u_5 & \text{if } C(u_3u_7) = i_4; \\ u_5u_4v_1u_6u_7u_3u_2u_1 & \text{if } C(u_3u_7) = i_5. \end{cases}$$

Then, P' is a heterochromatic path of length 7, a contradiction.

Case 2.2 There exists a j_0 such that $1 \leq j_0 \leq k-l-1$, and $y_{j+1} = y_j + 2$ for any $1 \leq j \leq k-l-1$ and $j \neq j_0$, $y_{j_0+1} = y_{j_0} + 3$.

In this case, we have $CN(u_{l+1}) - C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}, u_lu_{l+1}\}) \subseteq C(P) - \{i_{y_1}, i_{y_1+1}, \ldots, i_{y_{j_0}-1}, i_{y_{j_0+1}}, i_{y_{j_0+1}+1}, \ldots, i_{y_{k-l}-1}\}$ by Lemma 3.2 and the fact that P is the longest heterochromatic path in G. Note that $y_{k-l} \leq l$, and so $i_l \notin \{i_{y_1}, i_{y_1+1}, i_{y_2}, i_{y_2+1}, \ldots, i_{y_{k-l}-1}\}$. Then $CN(u_{l+1}) - C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}\}) \subseteq C(P) - \{i_{y_1}, i_{y_1+1}, \ldots, i_{y_{j_0}-1}, i_{y_{j_0+1}}, i_{y_{j_0+1}+1}, \ldots, i_{y_{k-l}-1}\}$.

On the other hand, $CN(u_{l+1}) - C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}\}) \subseteq C(P) - \{i_{x_1-1}, i_{x_2-1}, \ldots, i_{x_{k-l}-1}\}$. Therefore $(CN(u_{l+1}) - C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}\})) \subseteq C(P) - \{i_{y_1}, i_{y_1+1}, \ldots, i_{y_{j_0}-1}, i_{y_{j_0+1}}, i_{y_{j_0+1}+1}, \ldots, i_{y_{k-l}-1}\} \cup \{i_{x_1-1}, i_{x_2-1}, \ldots, i_{x_{k-l}-1}\}$.

Note that $3 \leq x_1 < x_1 + 1 < x_2 < \ldots < x_{k-l} \leq l$, $|\{x_1, x_2, \ldots, x_{k-l}\} - \{y_1+1, y_2, y_2+1, \ldots, y_{j_0}\} \cup \{y_{j_0+1}+1, y_{j_0+2}, \ldots, y_{k-l}\}| \geq 2$. So $|\{i_{x_1-1}, i_{x_2-1}, \ldots, i_{x_{k-l}-1}\} - \{i_{y_1}, i_{y_1+1}, \ldots, i_{y_{j_0}-1}, i_{y_{j_0+1}}, i_{y_{j_0+1}+1}, \ldots, i_{y_{k-l}-1}\}| \geq 2$.

Consequently, $l-1 \geq |C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}\})| \geq k - |C(P) - \{i_{x_1-1}, i_{x_2-1}, \ldots, i_{x_{k-l}-1}\} \cup \{i_{y_1}, i_{y_1+1}, \ldots, i_{y_{j_0}-1}, i_{y_{j_0+1}}, i_{y_{j_0+1}+1}, \ldots, i_{y_{k-l}-1}\}| \geq k-$

$l + 2(j_0 - 1) + 2(k - l - j_0 - 1) + 2 = 3k - 3l - 2$. It is easy to check that if $k > 11$, $3k - 3l - 2 > l - 1$, and so we need only to consider the cases when $k = 8$ or $k = 11$.

Case 2.2.1 $k = 8$. In this case, $y_1 = 3$ and $y_2 = 6$. Denote $i_7 = C(u_3v_1)$ and $i_8 = C(u_6v_1)$. We distinguish the following cases:

Case 2.2.1.1 $x_1 = 3$ and $x_2 = 5$. Then $|C(u_1u_7, u_2u_7, u_3u_7, u_4u_7, u_5u_7) - \{i_1, i_3, i_5, i_6\}| \geq 4$.

If $C(u_1u_5) \notin \{i_7, i_8\}$, then $u_4u_5u_1u_2u_3v_1u_6u_7$ is a heterochromatic path of length 7, a contradiction. So we may assume $C(u_1u_5) \in \{i_7, i_8\}$.

If $C(u_1u_7) = i_2$, then $v_1u_3u_4u_5u_6u_7u_1u_2$ is a heterochromatic path of length 7, a contradiction.

If $C(u_1u_7) = i_4$, then since $|C(u_1u_7, u_2u_7, u_3u_7, u_4u_7, u5u_7) - \{i_1, i_3, i_5, i_6\}| \geq 4$, we have $C(u_2u_7, u_3u_7, u_4u_7, u_5u_7) - \{i_1, i_3, i_4, i_5, i_6, i_7, i_8\} \neq \emptyset$. Let

$$P' = \begin{cases} v_1u_3u_4u_5u_6u_7u_2u_1 & \text{if } C(u_2u_7) \notin \{i_1, i_3, i_4, i_5, i_6, i_7, i_8\}; \\ u_5u_6v_1u_3u_4u_7u_1u_2 & \text{if } C(u_4u_7) \notin \{i_1, i_3, i_4, i_5, i_6, i_7, i_8\}; \\ u_4u_3v_1u_6u_5u_7u_1u_2 & \text{if } C(u_5u_7) \notin \{i_1, i_3, i_4, i_5, i_6, i_7, i_8\}. \end{cases}$$

Then, P' is a heterochromatic path of length 7, a contradiction. So $C(u_3u_7) - \{i_1, i_3, i_4, i_5, i_6, i_7, i_8\} \neq \emptyset$. In this case, $u_2u_1u_5u_4u_3u_7u_6v_1$ is a heterochromatic path of length 7 if $C(u_1u_5) = i_7$. Now it remains to consider the case when $C(u_1u_5) = i_8$. Since $u_1u_2u_3v_1u_6u_5u_4$ is a heterochromatic path of length 6, $C(u_1u_3, u_1u_4, u_1u_6, u_1v_1) - \{i_1, i_2, i_4, i_5, i_6, i_7, i_8\} \neq \emptyset$. Let

$$P'' = \begin{cases} u_5u_4u_1u_2u_3v_1u_6u_7 & \text{if } C(u_1u_4) \notin \{i_1, i_2, i_4, i_5, i_6, i_7, i_8\}; \\ u_2u_3v_1u_7u_6u_1u_5u_4 & \text{if } C(u_1u_6) \notin \{i_1, i_2, i_4, i_5, i_6, i_7, i_8\}; \\ v_1u_1u_2u_3u_4u_5u_6u_7 & \text{if } C(u_1v_1) \notin \{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8\}; \\ u_2u_3u_1v_1u_7u_6u_5u_4 & \text{if } C(u_1v_1) = i_3. \end{cases}$$

Then, P'' is a heterochromatic path of length 7, and so $C(u_1u_3) \notin \{i_1, i_2, i_4, i_5, i_6, i_7, i_8\}$, i.e., $C(u_1u_3) \notin \{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8\}$. Denote $i_9 = C(u_1u_3)$. Since $C(u_1v_1, u_1u_4, u_1u_6) \subseteq \{i_1, i_2, \ldots, i_6, i_8, i_9\}$ and $d^c(u_1) \geq 8$, there exists a $v_2 \notin \{u_1, u_2, \ldots, u_7, v_1\}$ such that $C(u_1v_2) = i_3$. Then, $v_2u_1u_2u_3v_1u_6u_5u_4$ is a heterochromatic path of length 7, a contradiction.

If $C(u_1u_7) \neq i_4$, then $|C(u_2u_7, u_3u_7, u_4u_7, u_5u_7) - \{i_1, i_3, i_5, i_6\}| = 4$. Note that if $C(u_1u_7) = i_3$, then $u_4u_5u_6v_1u_3u_2u_1u_7$ is a heterochromatic path of length 7; if $C(u_1u_7) = i_5$, then $u_5u_4u_3v_1u_6u_7u_1u_2$ is a heterochromatic path of length 7. Then we can conclude that there exist vertices $v_2, v_3 \notin \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ such that $C(u_7v_2) = i_3$, $C(u_7v_3) = i_5$. If $C(u_1u_5) = i_8$, then $v_1u_3u_2u_1u_5u_6u_7v_2$ is a heterochromatic path of length 7, and so we assume $C(u_1u_5) = i_7$. Since $v_1u_6u_5u_1u_2u_3u_4$ is a heterochromatic path of length 6, we have $C(u_1v_1, u_2v_1, u_4v_1, u_5v_1) - \{i_1, i_2, i_3, i_5, i_6, i_7, i_8\} \neq \emptyset$. Let

$$P' = \begin{cases} u_4u_3u_2u_1v_1u_6u_7v_3 & \text{if } C(u_1v_1) \notin \{i_1, i_2, i_3, i_5, i_6, i_7, i_8\}; \\ u_4u_3u_2v_1u_7u_6u_5u_1 & \text{if } C(u_2v_1) \notin \{i_1, i_2, i_3, i_5, i_6, i_7, i_8\}; \\ v_1u_4u_3u_2u_1u_5u_6u_7 & \text{if } C(u_4v_1) \notin \{i_1, i_2, i_3, i_5, i_6, i_7, i_8\}; \\ u_4u_3u_2u_1u_5v_1u_6u_7 & \text{if } C(u_5v_1) \notin \{i_1, i_2, i_3, i_5, i_6, i_7, i_8\}. \end{cases}$$

Then, P' is a heterochromatic path of length 7, a contradiction.

Case 2.2.1.2 $x_1 = 3$ and $x_2 = 6$. Then $|C(u_1u_7, u_2u_7, u_3u_7, u_4u_7, u5u_7) - \{i_1, i_3, i_4, i_6\}| \geq 4$.

If $C(u_1u_7) = i_2$, then $v_1u_3u_4u_5u_6u_7u_1u_2$ is a heterochromatic path of length 7; if $C(u_1u_7) = i_5$, then $u_5u_4u_3v_1u_6u_7u_1u_2$ is a heterochromatic path of length 7, a contradiction. So we conclude that $|C(u_2u_7, u_3u_7, u_4u_7, u_5u_7) - \{i_1, i_3, i_4, i_6\}| = 4$. If $C(u_2u_7) \notin \{i_1, i_3, i_4, i_5, i_6, i_7\}$, then $v_1u_3u_4u_5u_6u_7u_2u_1$ is a heterochromatic path of length 7; if $C(u_2u_7) = i_5$, then $u_5u_4u_3v_1u_6u_7u_2u_1$ is a heterochromatic path of length 7, a contradiction. So $C(u_2u_7) = i_7$. Since $u_1u_2u_3v_1u_6u_5u_4$ is a heterochromatic path of length 6, we have $C(u_1u_3, u_1u_4, u_1u_5, u_1u_6, u_1v_1) - \{i_1, i_2, i_3, i_4, i_5, i_7, i_8\} \neq \emptyset$. Let

$$P' = \begin{cases} u_1u_3u_2u_7v_1u_6u_5u_4 & \text{if } C(u_1u_3) \notin \{i_1, i_2, i_3, i_4, i_5, i_7, i_8\}; \\ u_1u_4u_3u_2u_7v_1u_6u_5 & \text{if } C(u_1u_4) \notin \{i_1, i_2, i_3, i_4, i_5, i_7, i_8\}; \\ u_1u_5u_4u_3u_2u_7v_1u_6 & \text{if } C(u_1u_5) \notin \{i_1, i_2, i_3, i_4, i_5, i_7, i_8\}; \\ u_1u_6u_5u_4u_3u_2u_7v_1 & \text{if } C(u_1u_6) \notin \{i_1, i_2, i_3, i_4, i_5, i_7, i_8\}; \\ u_1v_1u_7u_2u_3u_4u_5u_6 & \text{if } C(u_1v_1) \notin \{i_1, i_2, i_3, i_4, i_5, i_7, i_8\}. \end{cases}$$

Then, P' is a heterochromatic path of length 7, a contradiction.

Case 2.2.1.3 $x_1 = 4$ and $x_2 = 6$. Then $|C(u_1u_7, u_2u_7, u_3u_7, u_4u_7, u5u_7) - \{i_1, i_2, i_4, i_6\}| \geq 4$.

If $C(u_1u_7) = i_3$, then $v_1u_3u_2u_1u_7u_6u_5u_4$ is a heterochromatic path of length 7; if $C(u_1u_7) = i_5$, then $v_1u_6u_7u_1u_2u_3u_4u_5$ is a heterochromatic path of length 7, a contradiction. So we get that $|C(u_2u_7, u_3u_7, u_4u_7, u_5u_7) - \{i_1, i_2, i_4, i_6\}| = 4$. If $C(u_1u_4) \neq i_7$, then $v_1u_3u_2u_1u_4u_5u_6u_7$ is a heterochromatic path of length 7, and so $C(u_1u_4) = i_7$ and $C(u_1u_6) \notin \{i_1, i_2, i_3, i_4, i_5, i_6, i_7\}$. If $C(u_1u_7) = i_2$, then $v_1u_3u_4u_5u_6u_7u_1u_2$ is a heterochromatic path of length 7, and so there exists a $v_2 \notin \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ such that $C(u_7v_2) = i_2$. Then, $u_1u_6u_5u_4u_3v_1u_7v_2$ is a heterochromatic path of length 7, a contradiction.

So, in the case $k = 8$, there exists a heterochromatic path of length 7 in G.

Case 2.2.2 $k = 11$. Denote $i_9 = C(u_{y_1}v_1)$, $i_{10} = C(u_{y_2}v_1)$ and $i_{11} = C(u_{y_3}v_1)$. We distinguish the following two cases:

Case 2.2.2.1 $y_1 = 3$, $y_2 = 6$ and $y_3 = 8$.

We can easily get that $x_3 = 7$ or $x_3 = 8$. Since $3k - 3l - 2 = l - 1$ in this case, we have $|C(u_1u_9, u_2u_9, \ldots, u_6u_9, u_7u_9) - (\{i_1, i_2, i_3, i_4, i_5, i_8\} - \{i_{x_1-1}, i_{x_2-1}\})| = 7$. Then, $C(u_1u_9) \in \{i_{x_1-1}, i_{x_2-1}, i_6, i_7\}$. Let

$$P' = \begin{cases} v_1u_3u_4u_5u_6u_7u_8u_9u_1u_2 & \text{if } C(u_1u_9) = i_2; \\ u_4u_5u_6u_7u_8u_9u_1u_2u_3v_1 & \text{if } C(u_1u_9) = i_3; \\ v_1u_6u_7u_8u_9u_1u_2u_3u_4u_5 & \text{if } C(u_1u_9) = i_5; \\ u_7u_8u_9u_1u_2u_3u_4u_5u_6v_1 & \text{if } C(u_1u_9) = i_6; \\ v_1u_8u_9u_1u_2u_3u_4u_5u_6u_7 & \text{if } C(u_1u_9) = i_7. \end{cases}$$

Then, P' is a heterochromatic path of length 9, a contradiction. So $C(u_1u_9) = i_4$, and then we can conclude that $5 \in \{x_1, x_2\}$ and $4 \notin \{x_1, x_2\}$. Therefore, there exists a $v_2 \notin \{u_1, u_2, \ldots, u_8, u_9\}$ such that $C(u_9v_2) = i_3$, and $u_5u_6u_7u_8v_1u_3u_2u_1u_9v_2$ is a heterochromatic path of length 9, a contradiction.

Case 2.2.2.2 $y_1 = 3$, $y_2 = 5$ and $y_3 = 8$.

Since $3k-3l-2 = l-1$ in this case, we have $|\{i_1, i_2, i_5, i_6, i_7, i_8\} \cap \{i_{x_1-1}, i_{x_2-1}, i_{x_3-1}\}| = 2$ and $|C(u_1u_9, u_2u_9, \ldots, u_6u_9, u_7u_9) - (\{i_1, i_2, i_5, i_6, i_7, i_8\} - \{i_{x_1-1}, i_{x_2-1}, i_{x_3-1}\})| = 7$. Then we can get that $x_1 = 3, x_2 = 5$ and $x_3 = 7$, or $x_1 = 3, x_2 = 5$ and $x_3 = 8$, or $x_1 = 4, x_2 = 6$ and $x_3 = 8$, and $C(u_1u_9) \in \{i_3, i_4, i_{x_1-1}, i_{x_2-1}, i_{x_3-1}\}$. Let

$$P' = \begin{cases} v_1u_3u_4u_5u_6u_7u_8u_9u_1u_2 & \text{if } C(u_1u_9) = i_2; \\ u_4u_5u_6u_7u_8u_9u_1u_2u_3v_1 & \text{if } C(u_1u_9) = i_3; \\ v_1u_5u_6u_7u_8u_9u_1u_2u_3u_4 & \text{if } C(u_1u_9) = i_4; \\ u_6u_7u_8u_9u_1u_2u_3u_4u_5v_1 & \text{if } C(u_1u_9) = i_5; \\ v_1u_8u_9u_1u_2u_3u_4u_5u_6u_7 & \text{if } C(u_1u_9) = i_7. \end{cases}$$

Then, P' is a heterochromatic path of length 9, a contradiction. So $C(u_1u_9) = i_6$, and then $x_1 = 3, x_2 = 5, x_3 = 7$. Therefore, there exists a $v_2 \notin \{u_1, u_2, \ldots, u_8, u_9\}$ such that $C(u_9v_2) = i_5$, and $u_7u_8v_1u_5u_4u_3u_2u_1u_9v_2$ is a heterochromatic path of length 9, a contradiction.

So, in the case $k = 11$, there exists a heterochromatic path of length 9 in G.

Up to now, we can conclude that if $d^c(v) \geq k \geq 7$ for any $v \in V(G)$, then G has a heterochromatic path of length at least $\lceil \frac{2k}{3} \rceil + 1$ in G. ∎

3. Long heterochromatic paths under the color neighborhood union condition

Let G be an edge-colored graph and s a positive integer. Suppose that $|CN(u) \cup CN(v)| \geq s$ for every pair of vertices u and v of G. It is easy to see that if $s = 1, 2$ then G has a heterochromatic path of length s, and if $s = 3$ then G has a heterochromatic path of length 2. In [4], the authors showed that G has a heterochromatic path of length at least $\lceil \frac{s}{3} \rceil + 1$ for $s > 1$. In this section we will improve this lower bound for $s \geq 4$.

Theorem 3.1 *Let G be an edge-colored graph and s a positive integer. Suppose that $|CN(u) \cup CN(v)| \geq s \geq 4$ for every pair of vertices u and v of G. Then G has a heterochromatic path of length at least $\lfloor \frac{2s+4}{5} \rfloor$.*

Proof. By contradiction. Suppose $P = u_1u_2 \ldots u_lu_{l+1}$ is a longest heterochromatic path of length $l < \lfloor \frac{2s+4}{5} \rfloor$. Denote $i_j = C(u_ju_{j+1})$ for $j = 1, 2, \ldots, l$.

Since P is a longest heterochromatic path in G, there exist x_i's and y_j's such that $3 \leq x_1 < x_2 < \ldots < x_{t_1} \leq l+1$ and $2 \leq y_1 < y_2 < \ldots < y_{t_2} \leq l-1$, and $t_1 = |CN(u_1) - C(P)| = |C(u_1u_{x_1}, u_1u_{x_2}, \ldots, u_1u_{x_{t_1}})|$, $t_2 = |CN(u_{l+1}) - C(P)| = |C(u_{y_1}u_{l+1}, u_{y_2}u_{l+1}, \ldots, u_{y_{t_2}}u_{l+1})|$ and $C(u_1u_{x_1}, u_1u_{x_2}, \ldots, u_1u_{x_{t_1}}) \cap C(u_{y_1}u_{l+1}, u_{y_2}u_{l+1}, \ldots, u_{y_{t_2}}u_{l+1}) = \emptyset$. Then $t_1 + t_2 \geq s - l > s - \lfloor \frac{2s+4}{5} \rfloor = \lceil \frac{3s-4}{5} \rceil \geq \frac{3s-4}{5} > \frac{2s-1}{5} \geq \lfloor \frac{2s+4}{5} \rfloor - 1 > l - 1$. Denote $\{z_1, z_2, \ldots, z_{t_3}\} = \{y_1, y_2, \ldots, y_{t_2}\} \cap \{x_1 - 1, x_2 - 1, \ldots, x_{t_1} - 1\}$, and so $2 \leq z_1 < z_2 < \ldots < z_{t_3} \leq l - 1$. Since

$2 \leq y_1 < y_2 < \ldots < y_{t_2} \leq l-1$ and $2 \leq x_1 - 1 < x_2 - 1 < \ldots < x_{t_1} - 1 \leq l$, we have $t_3 \geq t_1 + t_2 - (l-1) > 0$. Then, from Lemma 2.2 we can get that $CN(u_1) - C(u_1u_3, u_1u_4, \ldots, u_1u_l, u_1u_{l+1}) \subseteq C(P) - \{i_{z_1}, i_{z_2}, \ldots, i_{z_3}\}$, $CN(u_{l+1}) - C(u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}) \subseteq C(P) - \{i_{z_1}, i_{z_2}, \ldots, i_{z_3}\}$. So, $CN(u_1) \cup CN(u_{l+1}) \subseteq (C(P) - \{i_{z_1}, i_{z_2}, \ldots, i_{z_3}\}) \cup C(u_1u_3, u_1u_4, \ldots, u_1u_l, u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1})$. Therefore, $|CN(u_1) \cup CN(u_{l+1})| \leq |C(P) - \{i_{z_1}, i_{z_2}, \ldots, i_{z_3}\}| + |C(u_1u_3, u_1u_4, \ldots, u_1u_l, u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1})| = (l - t_3) + (2l-3) = 3l - 3 - t_3 \leq 3l - 3 - (t_1 + t_2) + (l-1) = 4l - 4 - (t_1 + t_2) \leq 4l - 4 - (s-l) = 5l - 4 - s < 5 * \lfloor \frac{2s+4}{5} \rfloor - 4 - s \leq s$, a contradiction.

So, if $|CN(u) \cup CN(v)| \geq s \geq 4$ for every pair of vertices u and v of G, then G has a heterochromatic path of length at least $\lfloor \frac{2s+4}{5} \rfloor$. ∎

Although we cannot show that the above lower bound is best possible, the following example shows that the best lower bound cannot be better than $\lfloor \frac{s}{2} \rfloor + 1$. Let s be a positive integer. If s is even, let G_s be the graph obtained from the complete graph $K_{\frac{s+4}{2}}$ by deleting an edge; if s is odd, let G_s be the complete graph $K_{\frac{s+3}{2}}$. Then, color the edges of G_s by different colors for any two different edges. So, for any $s \geq 1$ we have that $|CN(u) \cup CN(v)| \geq s$ for any pair of vertices u and v in G, and any longest heterochromatic path in G is of length $\lfloor \frac{s}{2} \rfloor + 1$. This example shows that the lower bound in our Theorem 4.1 is not very far away from the best.

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